

# SU(1,2) interferometer

Yadong Wu\*

UM-SJTU Joint Institute, Shanghai Jiao Tong University, Shanghai, 200240, PR China

Chun-Hua Yuan†

Quantum Institute for Light and Atoms, Department of Physics,  
East China Normal University, Shanghai 200062, P. R. China

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We theoretically investigate an interferometer composed of four four-wave-mixers by Lie group method. Lie group  $SU(1,2)$  characterizes the mode transformations of this kind of interferometer. With vacuum state inputs, the phase sensitivity of  $SU(1,2)$  interferometer achieves the Heisenberg limit, and the absolute accuracy beats  $SU(1,1)$  interferometer because of higher intensity of light inside the interferometer. For different input cases, the optimal combination of output photon number for detection to obtain the best phase sensitivity is calculated. Our research on  $SU(1,2)$  interferometer sheds light on the performance of  $SU(1,n)$  interferometer in quantum metrology.

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## I. INTRODUCTION

Quantum metrology [1, 2] takes the advantages of quantum mechanics to realize the high resolution of parameter measurement in physical systems. The original motivation of studying quantum metrology comes from general relativity. General relativity predicts the existence of gravitational wave, which, however, is too weak to be experimentally detected. To enhance the sensitivity of interferometers, physicists feed high power laser into interferometers. But it seems that the sensitivity of this kind of laser interferometers is restricted by standard quantum limit (SQL) [3]. SQL indicates that the phase sensitivity is bounded by  $1/\sqrt{n}$ , where  $n$  is the photon number. The best phase sensitivity which can be achieved with classical interferometer, like Mach-Zehnder interferometer (MZI) powered by a coherent state laser, cannot go beyond the SQL. But SQL is not the ultimate limit physicists can achieve, they develop new methods to beat the SQL and achieve the Heisenberg limit, which indicates that the phase sensitivity equals to  $1/n$ , i.e., the reciprocal of the photon number. This kind of interferometers beating SQL are quantum interferometers.

With the development of modern mathematics, Lie group theory is widely used to describe symmetric structures in theoretical physics. In [4], Lie group theory is used to describe the mode transformations in an interferometer. Specifically, the interferometers like MZI, consisting of beam splitters, can be characterized by the Lie group  $SU(2)$ , while the interferometer using four wave mixers to replace beam splitters is characterized by the Lie group  $SU(1,1)$ . This Lie group structure is determined by the Lie algebra formed by the Hamiltonian of the interferometer. With single Fock state input, the

$SU(2)$  interferometer cannot go beyond the SQL. But fed with a special correlated Fock state  $(|0,0\rangle + |0,2\rangle)/\sqrt{2}$ ,  $SU(2)$  interferometer can attain the Heisenberg limit [4]. Besides, it is shown that if the second input of a MZI is fed with a squeezed vacuum state, the phase sensitivity can beat the SQL [3]. In contrast,  $SU(1,1)$  interferometer can achieve the Heisenberg limit with vacuum state inputs. It will be much more easier to implement since it needs fewer optical devices experimentally. But since the photon number inside the interferometer with vacuum state inputs is relatively low, W. N. Plick et al. [5] suggest injecting coherent state light into  $SU(1,1)$  interferometer to significantly enhance the phase sensitivity. Feeding strong coherent state laser makes the photon number in  $SU(1,1)$  interferometer increased vastly and makes the phase sensitivity reaching far below the SQL. However, the phase sensitivity can only inversely scale with the square root of the intensity of the coherent state input light. D. Li et al. [6] proposes that injecting a coherent state and a squeezed vacuum state into  $SU(1,1)$  interferometer with homodyne detection can decrease the noise and enhance the sensitivity. But this method cannot always attain the Heisenberg limit, either. In this paper, we present a new interferometer called  $SU(1,2)$  interferometer composed of four four-wave-mixers (FWMs) which is described by the Lie group  $SU(1,2)$ . The absolute accuracy can not only beat the sensitivity of  $SU(1,1)$  interferometer for the same input fields, but also make the phase sensitivity scale with the Heisenberg limit.

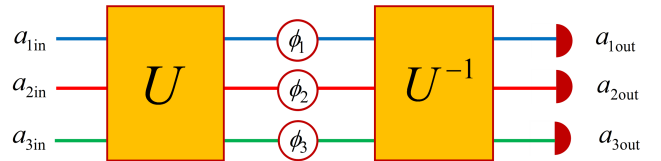


FIG. 1: (Color online) General three-port balanced interferometer.  $\phi_i$  ( $i = 1, 2, 3$ ): phase shifts.

\*Electronic address: wuyadong301@sjtu.edu.cn

†Electronic address: chuyuan@phy.ecnu.edu.cn

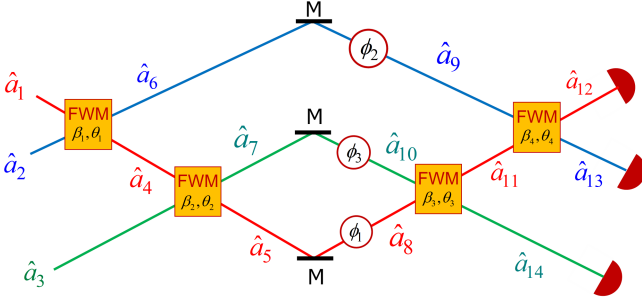


FIG. 2: (Color online) The schematic diagram of the  $SU(1,2)$  interferometer.  $\hat{a}_i$  ( $i = 0, 1, \dots, 14$ ) denote the different light beams in the interferometer.  $\beta_i$  and  $\theta_i$  ( $i = 1, 2, 3, 4$ ) describe the strength and phase shift in the different four-wave-mixers (FWMs) processes.  $\phi_i$  ( $i = 1, 2, 3$ ): phase shifts;  $M$ : mirrors.

These interferometers we mentioned are two input and two output interferometers, whereas three input and three output  $SU(3)$  interferometer [7–9] and other multi-path interferometers [10–14] are also investigated for different purposes. Multi-path interferometer generally owns more than two input and two output ports. Over two beams are mixed in a multi-path interferometer. B. C. Sanders et al. [7] investigates measuring vector phase with  $SU(3)$  interferometer.  $SU(3)$  interferometer, different from a two-path interferometer, can be used to estimate two phase shifts simultaneously. G. M. D’Ariano et al. [11] propose a scheme of using multi-path interferometer to enhance phase sensitivity and asserts that the phase sensitivity can inversely scale with the number of beam paths in an interferometer. It seems that there are two advantages of multi-path interferometers in phase estimation. One is to estimate more phases simultaneously, the other is to enhance the phase sensitivity. These two advantages motivate us to investigate quantum metrology with multi-path interferometer.

As far as we know, most multi-path interferometers discussed previously uses linear optical devices like beam splitters, while multi-path nonlinear quantum interferometer utilizing nonlinear optical devices such as FWM has not been theoretically investigated. We want to use the Lie group theoretical method in [4] to investigate a multi-path nonlinear quantum interferometer, denoted by  $SU(1,2)$  interferometer. We’re interested in the sensitivity of  $SU(1,2)$  interferometer in phase estimation and want to compare its phase sensitivity with  $SU(1,1)$  interferometer [4, 5].

This paper mainly contains three parts. Sec. II introduces eight Hermitian operators spanning the Lie algebra  $\mathfrak{su}(1,2)$ . The mode transformations given by the conjugation operation form the group  $SU(1,2)$ . Then we introduce the structure of  $SU(1,2)$  interferometer. Sec. III shows that with vacuum state inputs,  $SU(1,2)$  interferometer can attain the Heisenberg limit and beats the sensitivity of  $SU(1,1)$  interferometer. The optimal combination of output photon numbers for detection is found to achieve the best sensitivity. Sec. IV analyzes the case that a coherent state beam is fed into  $SU(1,2)$  inter-

ferometer. We give the optimal combination of output photon numbers for detection in different input cases. Furthermore, if the coherent state light is injected into the third input, the phase sensitivity can still approach the Heisenberg limit when the amplification gain of the FWMs is large. Sec. V gives the conclusion of this paper.

## II. $SU(1,2)$ LIE GROUP AND $SU(1,2)$ INTERFEROMETER

In this section, based on the method of Lie group we introduce the  $SU(1,2)$  interferometer possessing three inputs and three outputs. A general three input and three output interferometer is shown in Fig. 1. If we restrict the unitary operation  $U$  to  $SU(1,2)$  operation, then it is a  $SU(1,2)$  interferometer. Then in Fig. 2 we use the non-linear optical devices FWMs to form one type of  $SU(1,2)$  interferometer. The connection between  $SU(1,2)$  Lie group and the optical devices FWMs are presented. It can be constructed with current experimental technology [15].

A  $SU(1,2)$  mode transformation is

$$\begin{pmatrix} \hat{a}_{1\text{out}} \\ \hat{a}_{2\text{out}}^\dagger \\ \hat{a}_{3\text{out}}^\dagger \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} \hat{a}_{1\text{in}} \\ \hat{a}_{2\text{in}}^\dagger \\ \hat{a}_{3\text{in}}^\dagger \end{pmatrix}, \quad (1)$$

where the creative and annihilation operators satisfy the boson commutation relations. The transformation matrix has the following relation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}^{-1}. \quad (2)$$

All the matrices satisfying Eq. (2) form a matrix Lie group  $SU(1,2)$ . To generate these  $SU(1,2)$  mode transformations, we introduce eight Hermitian operators which are given by

$$\begin{aligned} K_1 &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2), \quad K_2 = -\frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2), \\ K_3 &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_3^\dagger + \hat{a}_1 \hat{a}_3), \quad K_4 = -\frac{i}{2}(\hat{a}_1^\dagger \hat{a}_3^\dagger - \hat{a}_1 \hat{a}_3), \\ K_5 &= -\frac{1}{2}(\hat{a}_2^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_2), \quad K_6 = -\frac{i}{2}(\hat{a}_2^\dagger \hat{a}_3 - \hat{a}_3^\dagger \hat{a}_2), \\ K_7 &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2 \hat{a}_2^\dagger), \quad K_8 = \frac{1}{2\sqrt{3}}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2 \hat{a}_2^\dagger + 2\hat{a}_3 \hat{a}_3^\dagger). \end{aligned} \quad (3)$$

Each operator  $K_i$  ( $i = 1, 2, \dots, 8$ ) in Eqs. (3) is a Hamiltonian of some physical process generating  $SU(1,2)$  mode transformations.  $K_1, K_2, K_3$  and  $K_4$  create or annihilate

photons in pairs, which describe the process of generating photons in FWMs.  $K_1$  and  $K_2$  are the Hamiltonian of a FWM with beam 1 and beam 2 as inputs, and  $K_3$  and  $K_4$  are the Hamiltonian of a FWM with beam 1 and beam 3 as inputs.  $K_5$  and  $K_6$  annihilate a photon and create another photon simultaneously, maintaining the total photon number. It is a passive optical device, generally we call beam splitter.  $K_5$  and  $K_6$  are the Hamiltonian of a beam splitter with beam 2 and beam 3 as inputs.  $K_7$  and  $K_8$  are combinations of photon number operators and numbers. They can be the Hamiltonian of relative phase shifts.

Using Baker-Campbell-Hausdorff (BCH) formula, we can calculate the conjugation of  $\hat{a}_1, \hat{a}_2^\dagger$  and  $\hat{a}_3^\dagger$  given by the exponential of the operators in Eqs. (3), which are given by

$$\begin{aligned}
e^{i\alpha_1 K_1} M_A e^{-i\alpha_1 K_1} &= \begin{pmatrix} \cosh \frac{\alpha_1}{2} & -i \sinh \frac{\alpha_1}{2} & 0 \\ i \sinh \frac{\alpha_1}{2} & \cosh \frac{\alpha_1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} M_A, \\
e^{i\alpha_2 K_2} M_A e^{-i\alpha_2 K_2} &= \begin{pmatrix} \cosh \frac{\alpha_2}{2} & -\sinh \frac{\alpha_2}{2} & 0 \\ -\sinh \frac{\alpha_2}{2} & \cosh \frac{\alpha_2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} M_A, \\
e^{i\alpha_3 K_3} M_A e^{-i\alpha_3 K_3} &= \begin{pmatrix} \cosh \frac{\alpha_3}{2} & 0 & -i \sinh \frac{\alpha_3}{2} \\ 0 & 1 & 0 \\ i \sinh \frac{\alpha_3}{2} & 0 & \cosh \frac{\alpha_3}{2} \end{pmatrix} M_A, \\
e^{i\alpha_4 K_4} M_A e^{-i\alpha_4 K_4} &= \begin{pmatrix} \cosh \frac{\alpha_4}{2} & 0 & -\sinh \frac{\alpha_4}{2} \\ 0 & 1 & 0 \\ -\sinh \frac{\alpha_4}{2} & 0 & \cosh \frac{\alpha_4}{2} \end{pmatrix} M_A, \\
e^{i\alpha_5 K_5} M_A e^{-i\alpha_5 K_5} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\alpha_5}{2} & -i \sinh \frac{\alpha_5}{2} \\ 0 & -i \sin \frac{\alpha_5}{2} & \cos \frac{\alpha_5}{2} \end{pmatrix} M_A, \\
e^{i\alpha_6 K_6} M_A e^{-i\alpha_6 K_6} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\alpha_6}{2} & -\sinh \frac{\alpha_6}{2} \\ 0 & \sin \frac{\alpha_6}{2} & \cos \frac{\alpha_6}{2} \end{pmatrix} M_A, \\
e^{i\alpha_7 K_7} M_A e^{-i\alpha_7 K_7} &= \begin{pmatrix} e^{-i\frac{\alpha_7}{2}} & 0 & 0 \\ 0 & e^{i\frac{\alpha_7}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} M_A, \\
e^{i\alpha_8 K_8} M_A e^{-i\alpha_8 K_8} &= \begin{pmatrix} e^{-i\frac{\alpha_8}{2\sqrt{3}}} & 0 & 0 \\ 0 & e^{-i\frac{\alpha_8}{2\sqrt{3}}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_8}{\sqrt{3}}} \end{pmatrix} M_A,
\end{aligned} \tag{4}$$

where  $M_A = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \\ \hat{a}_3^\dagger \end{pmatrix}$ . The transformations are in the form of Eq. (1), and satisfy Eq. (2), implying that the transformations are all  $SU(1, 2)$  matrices. From Eqs. (4), we can see that  $e^{i\alpha_1 K_1}$  and  $e^{i\alpha_2 K_2}$  amplify beam 1 and beam 2 and that  $e^{i\alpha_3 K_3}$  and  $e^{i\alpha_4 K_4}$  amplify beam 1 and

beam 3. They represent the operations of FWMs.  $e^{i\alpha_5 K_5}$  and  $e^{i\alpha_6 K_6}$  generate a combination of beam 2 and beam 3, which is like a rotation in the two-dimensional space spanned by  $\hat{a}_2$  and  $\hat{a}_3$ . They represent the operations of beam splitters. Whereas  $e^{i\alpha_7 K_7}$  and  $e^{i\alpha_8 K_8}$  multiply  $\hat{a}_1, \hat{a}_2$  and  $\hat{a}_3$  by a unit complex number. This process only changes the phase of each mode.

The eight operators in Eqs. (3) span the Lie algebra  $\mathfrak{su}(1, 2)$ . From the Lie group theory [16] we know that the conjugation given by the exponential of a Lie algebra forms a Lie group. Specifically, in our case,  $K_i$  ( $i$  is from 1 to 8) form the Lie algebra  $\mathfrak{su}(1, 2)$ , the conjugation given by the exponential of  $K_i$  in the vector space spanned by  $\hat{a}_1, \hat{a}_2^\dagger$  and  $\hat{a}_3^\dagger$  form the Lie group  $SU(1, 2)$ .

It's interesting that the combination of photon number operators  $\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_3^\dagger \hat{a}_3$  commute with all the  $K_i$ . It means that throughout any  $SU(1, 2)$  mode transformation, the photon number difference  $\hat{n}_1 - \hat{n}_2 - \hat{n}_3$  remains the same. It makes sense because the FWMs generate photon pairs in beam 1 and beam 2, or in beam 1 and beam 3, while beam splitter remain the total photon number in beam 2 and beam 3 invariant. Going through any combination of these devices, the same amount of photons will be added into beam 1 and the combination of beam 2 and beam 3.

A  $SU(1, 2)$  interferometer consisting of four FWMs is shown in Fig. 2. The two inputs of a FWM are different: one is called probe beam, the other one is idler beam. We use beam 1 and beam 2 to denote the probe beam and the idler beam of the first FWM respectively. The two output beams are denoted by beam 4 and beam 6 :

$$\begin{aligned}
\hat{a}_4 &= \cosh \frac{\beta_1}{2} \hat{a}_1 + e^{-i\theta_1} \sinh \frac{\beta_1}{2} \hat{a}_2^\dagger, \\
\hat{a}_6 &= \cosh \frac{\beta_1}{2} \hat{a}_2 + e^{-i\theta_1} \sinh \frac{\beta_1}{2} \hat{a}_1^\dagger,
\end{aligned}$$

where  $\beta_1$  and  $\theta_1$  are the amplification gain and the phase parameter of the first FWM. Later we use  $\theta_j$  and  $\beta_j$  ( $j$  is from 1 to 4) to denote the phase parameters and the amplification parameters of each FWM from left to right. Then beam 4 is injected into the second FWM as the probe beam, and beam 3 injected as the idler beam. The two output beams are denoted by beam 5 and beam 7:

$$\begin{aligned}
\hat{a}_5 &= \cosh \frac{\beta_2}{2} \hat{a}_4 + e^{-i\theta_2} \sinh \frac{\beta_2}{2} \hat{a}_3^\dagger, \\
\hat{a}_7 &= \cosh \frac{\beta_2}{2} \hat{a}_3 + e^{-i\theta_2} \sinh \frac{\beta_2}{2} \hat{a}_4^\dagger.
\end{aligned}$$

After that, beam 5, 6 and 7 experience three independent phase shifts,  $\phi_1, \phi_2$  and  $\phi_3$ , respectively. We denote the three beams after phase shifting as beam 8 to beam 10. Then they pass through the third and the fourth FWMs. The transformations of the annihilation operators in each

beam can be written as

$$\begin{aligned}\hat{a}_{12} &= \cosh \frac{\beta_4}{2} \hat{a}_{11} + e^{-i\theta_4} \sinh \frac{\beta_4}{2} \hat{a}_9^\dagger, \\ \hat{a}_{13} &= \cosh \frac{\beta_4}{2} \hat{a}_9 + e^{-i\theta_4} \sinh \frac{\beta_4}{2} \hat{a}_{11}^\dagger, \\ \hat{a}_{11} &= \cosh \frac{\beta_3}{2} \hat{a}_8 + e^{-i\theta_3} \sinh \frac{\beta_3}{2} \hat{a}_{10}^\dagger, \\ \hat{a}_{14} &= \cosh \frac{\beta_3}{2} \hat{a}_{10} + e^{-i\theta_3} \sinh \frac{\beta_3}{2} \hat{a}_8^\dagger.\end{aligned}\quad (5)$$

In the end, one can detect the sum of the output photon numbers in beam 12 and 13 and the photon number in beam 14.

In the following, we are going to investigate the phase sensitivity of  $SU(1,2)$  interferometer. The phase sensitivity is defined by

$$(\Delta\phi_j)_{sn_{12}+tn_{13}+rn_{14}} = \frac{\Delta(s\hat{n}_{12}+t\hat{n}_{13}+r\hat{n}_{14})}{\left| \frac{\partial(s\hat{n}_{12}+t\hat{n}_{13}+r\hat{n}_{14})}{\partial\phi_j} \right|}, \quad (6)$$

where  $j$  can be 1, 2, or 3,  $s, t, r$  are real numbers, and  $\hat{n}_{12} = \hat{a}_{12}^\dagger \hat{a}_{12}$ , similar for  $\hat{n}_{13}$  and  $\hat{n}_{14}$ , denoting the photon numbers in beam 12, 13, 14. Phase sensitivity is the standard deviation of the estimation result of the phase shift. Choose the output photon number as the estimator. The phase sensitivity equals to the standard deviation of the estimator divided by the derivative of the mean value of the estimator with respect to the corresponding phase shift. Various estimators result in different phase sensitivities for the same estimation target.

Without loss of generality, we set the phase parameters of the FWMs  $\theta_1 = \theta_2 = 0$  and  $\theta_3 = \theta_4 = \pi$ . The  $\pi$  change of the phase parameters makes the third and fourth FWMs perform the inverse operations of the first and second FWMs, respectively.

### III. PHASE SENSITIVITY WITH VACUUM STATE INPUTS

This section investigates the phase sensitivity of  $SU(1,2)$  interferometer when all the three inputs are vacuum states.

The detailed calculation of the phase sensitivity is omitted. Here we only focus on analysis of the results. We find that the optimal sensitivity is achieved when the sum of the photon numbers in beam 12 and beam 14 is detected. The comparison of the phase sensitivity of  $SU(1,2)$  interferometer with  $SU(1,1)$  interferometer shows its advantage of enhancing phase sensitivity.

In  $SU(1,2)$  interferometer, each phase sensitivity depends on the amount of the phase shift in each beam. Each phase sensitivity achieves its minimum when all the three phase shifts vanish, when the output state approaches the input state. Fig. 3 shows how the phase sensitivity  $\Delta\phi_1$  varies with respect to the other two phase shifts. The impact of shifting  $\phi_3$  on the sensitivity  $\Delta\phi_1$

is much greater than shifting  $\phi_2$ . Because

$$\hat{a}_5 = \cosh \frac{\beta_1}{2} \cosh \frac{\beta_2}{2} \hat{a}_1 + \sinh \frac{\beta_1}{2} \cosh \frac{\beta_2}{2} \hat{a}_2^\dagger + \sinh \frac{\beta_2}{2} \hat{a}_3^\dagger, \quad (7)$$

$$\hat{a}_7 = \cosh \frac{\beta_1}{2} \sinh \frac{\beta_2}{2} \hat{a}_1^\dagger + \sinh \frac{\beta_1}{2} \sinh \frac{\beta_2}{2} \hat{a}_2 + \cosh \frac{\beta_2}{2} \hat{a}_3. \quad (8)$$

For large  $\beta_2$ ,  $\hat{a}_5$  is nearly equal to  $\hat{a}_7^\dagger$ . Thus shifting  $\phi_3$  with  $\delta\phi$  is nearly equivalent to shifting  $\phi_1$  with  $-\delta\phi$ , which significantly affect  $\Delta\phi_1$ .

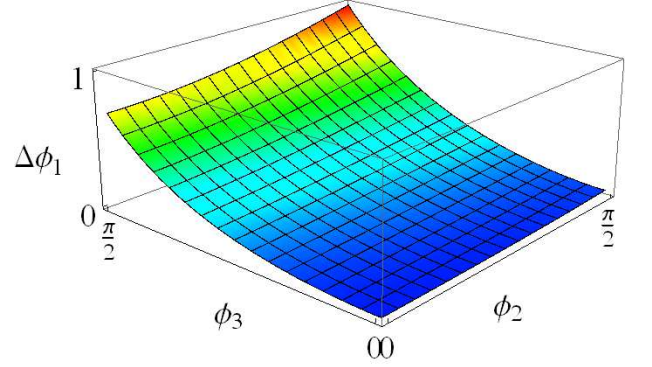


FIG. 3: (Color online) Phase sensitivity  $\Delta\phi_1$  is shown as a function of phase shifts  $\phi_2$  and  $\phi_3$ , when  $\beta_1 = \beta_2 = 3$ .

Since the phase shift can be controlled by feedback loops [17] to keep the phase sensitivity optimal, we assume during each single phase estimation, the other two phase shifts vanish. We will focus on the phase sensitivity  $\Delta\phi_1$  or  $\Delta\phi_3$ . As for  $\Delta\phi_2$ , when  $\phi_1 = \phi_3 = 0$ , the  $SU(1,2)$  interferometer is similar to the  $SU(1,1)$  interferometer, but the light intensities insider the interferometers are different.

At the output side, detecting different combinations of photon numbers in three outputs may lead to different phase sensitivities, among which, one combination is optimal. Fig. 4 shows the phase sensitivity  $\Delta\phi_1$  as a function of the ratios  $t/s$  and  $r/s$ . Any combination of the output photon numbers can be written as a combination of the linearly independent operators  $\hat{n}_{12} + \hat{n}_{13}$ ,  $\hat{n}_{12} - \hat{n}_{13} + 2\hat{n}_{14}$  and  $\hat{n}_{12} - \hat{n}_{13} - \hat{n}_{14}$ . The first two operators can be obtained from  $K_7$  and  $K_8$  at the output side, and the last one is the invariant operator equaling to  $\hat{n}_1 - \hat{n}_2 - \hat{n}_3$ . For vacuum state input,  $\langle \hat{n}_{12} - \hat{n}_{13} - \hat{n}_{14} \rangle = \text{Var}(\hat{n}_{12} - \hat{n}_{13} - \hat{n}_{14}) = 0$ . So we only need to find the optimal ratio between  $\hat{n}_{12} + \hat{n}_{13}$  and  $\hat{n}_{12} - \hat{n}_{13} + 2\hat{n}_{14}$ . It's found that the best combination is the sum of the output photon numbers in beam 12 and beam 14, i.e.,  $\hat{n}_{12} + \hat{n}_{14}$ , which is achieved when  $r/s = 1$  and  $t/s = 0$  in Fig. 4.

By taking the limitations of the expression of phase sensitivity at zero phase shifts, we obtain the optimal phase sensitivity  $(\Delta\phi_1)_{n_{12}+n_{14}}$  which is given by

$$(\Delta\phi_1)_{n_{12}+n_{14}} \sim \frac{2}{\cosh \beta_1 \cosh \beta_2}. \quad (9)$$



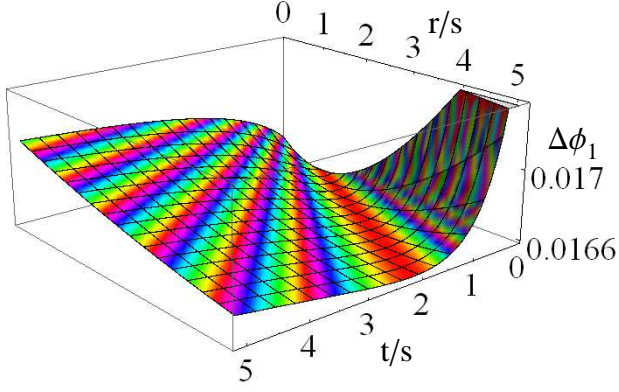


FIG. 4: (Color online) The phase sensitivity  $(\Delta\phi_1)_{sn_{12}+tn_{13}+rn_{14}}$  is plotted as a function of parameter ratios  $t/s$  and  $r/s$ . Set  $\beta_1 = \beta_2 = 3$ .

$(\Delta\phi_1)_{n_{12}+n_{13}}$  and  $(\Delta\phi_3)_{n_{12}+n_{13}}$  are not exactly the same, but when  $\cosh\beta_1, \cosh\beta_2 \gg 1$ , they are nearly equal to each other. The phase sensitivity of  $SU(1,1)$  interferometer is  $1/\sinh\beta$ , where  $\beta$  is the amplification gain of the FWMs in  $SU(1,1)$  interferometer. Because of a second optical parametric amplification, the phase sensitivity  $(\Delta\phi_1)_{n_{12}+n_{14}}$  and  $(\Delta\phi_3)_{n_{12}+n_{14}}$  are improved compared to the  $SU(1,1)$  interferometer. With vacuum states input, the absolute accuracy of  $SU(1,2)$  interferometer beats that of  $SU(1,1)$  interferometer.

In order to know whether the phase sensitivity can achieve the Heisenberg limit, we need to find how the phase sensitivities scale with the photon number. The total photon number in the interferometry is the sum of the photon numbers in beam 5, beam 6 and beam 7 as shown in Fig. 2, which is given by

$$N_{\text{total}} = \sinh^2 \frac{\beta_1}{2} (\cosh^2 \frac{\beta_2}{2} + 1) + \sinh^2 \frac{\beta_2}{2} (\cosh^2 \frac{\beta_1}{2} + 1). \quad (10)$$

The phase sensitivities  $(\Delta\phi_1)_{n_{12}+n_{14}}$  (solid red line) and  $(\Delta\phi_3)_{n_{12}+n_{14}}$  (dot-dashed green curve) as functions of the total photon number  $N_{\text{total}}$  are shown in Fig. 5. The dashed blue line represents the Heisenberg limit. It can be seen that both  $(\Delta\phi_1)_{n_{12}+n_{14}}$  and  $(\Delta\phi_3)_{n_{12}+n_{14}}$  nearly equal to the Heisenberg limit.

#### IV. PHASE SENSITIVITY WITH COHERENT STATE INPUT

This section still investigates the  $SU(1,2)$  interferometer in Fig. 2. Whereas the inputs become one coherent state  $|\alpha\rangle$  and two vacuum states. There are three different cases since a coherent state can be injected into any one of the three input ports. We will find that with one coherent state input, the phase sensitivity of  $SU(1,2)$  interferometer can be further enhanced and it's still possible for the phase sensitivity to achieve the Heisenberg limit.

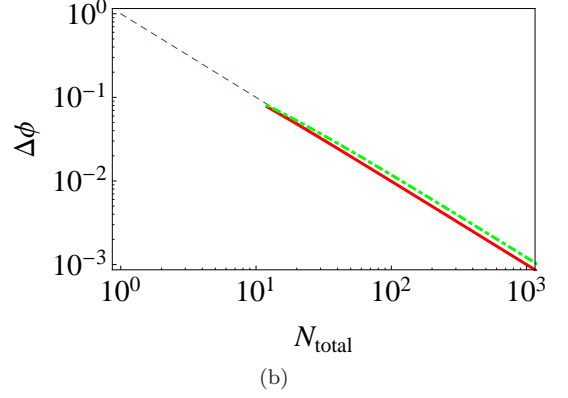
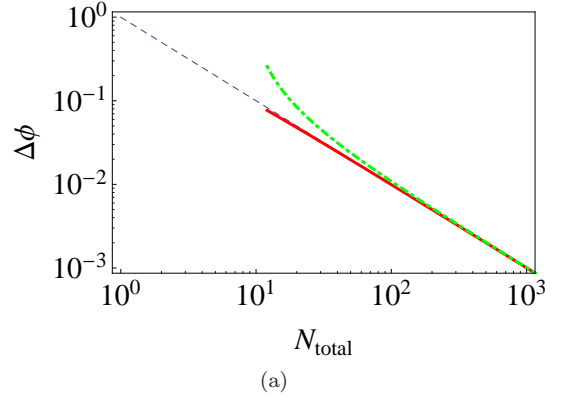


FIG. 5: (Color online) The phase sensitivities  $\Delta\phi$  of the  $SU(1,1)$  interferometer versus the total photon number  $N_{\text{total}}$  with parameter (a)  $\beta_1 = 3$ ; (b)  $\beta_2 = 3$ .  $(\Delta\phi_1)_{n_{12}+n_{14}}$  and  $(\Delta\phi_3)_{n_{12}+n_{14}}$  are described by the solid red line and the dot-dashed green curve, respectively. The dashed blue line is the Heisenberg limit.

When a coherent state beam is injected into input 1, to achieve the optimal phase sensitivity at zero phase shifts, one cannot choose the photon number  $n_{12}$  as the estimator to estimate any phase shift, otherwise the phase sensitivity will diverge. So  $s = 0$  in Eq. 6 and we can only detect the combination of photon numbers in beam 13 and beam 14. Which linear combination of  $n_{13}$  and  $n_{14}$  leads to the optimal phase sensitivity depends on the amplification gain of the FWMs and the intensity of the coherent state light input. Fig. 6 shows the optimal value of  $r/t$  to obtain the best phase sensitivity  $\Delta\phi_1$  when detecting the combination  $tn_{13} + rn_{14}$ . When the intensity of the coherent state input is low or the amplification gain of the FWMs is high, the best combination is  $n_{13} + n_{14}$ . Otherwise, the optimal  $r$  will be larger than 1. If the coherent state light is injected into the input 2, then one need to detect the combination of photon numbers in beam 12 and beam 14. It's similar for the case that the coherent state is in input 3. Fig. 7 shows the optimal ratio  $t/s$  when a coherent state light is fed into input 3.

Lastly, let's see whether the phase sensitivity of  $SU(1,2)$  interferometer with one coherent state input can

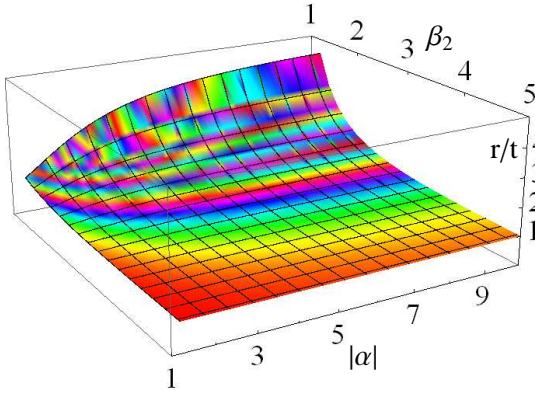


FIG. 6: (Color online) The optimal value of  $r/t$  as a function of the amplification gain of FWMs  $\beta_2$  and the intensity of the coherent state input  $|\alpha|$  is plotted.

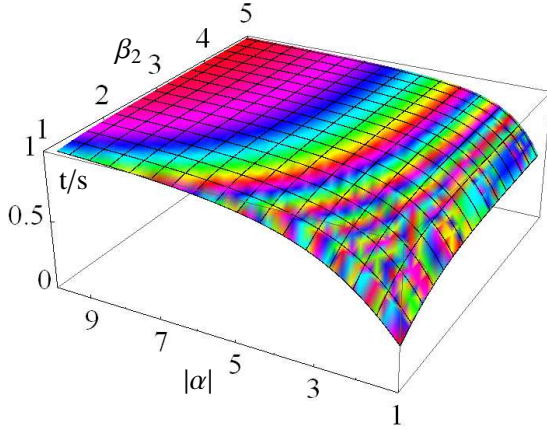


FIG. 7: (Color online) The optimal value of  $t/s$  as a function of the amplification gain of FWMs  $\beta_2$  and the intensity of the coherent state input  $|\alpha|$  is plotted.

achieve the Heisenberg limit. In the case of one coherent state light input, the total photon number will be different. When the coherent state light is fed into input 1 or input 3, the phase sensitivity  $(\Delta\phi_1)_{n_{13}+n_{14}}$  and  $(\Delta\phi_1)_{n_{12}+n_{13}}$  are respectively shown in Fig. 8 as a function of photon number  $N_{total}$ . It's shown that if the coherent state light is injected into the first input, the phase sensitivity cannot achieve the Heisenberg limit, while if the coherent state light is fed into the third input, it can attain the Heisenberg limit when the amplification gains are large. Comparing the values of the phase sensitivities in these two cases, we find injecting coherent state light into the third input leads to better phase sensitivity.

## V. DISCUSSION AND CONCLUSION

This paper discusses the ideal model of  $SU(1,2)$  interferometer, since photon number loss error has not been considered [18]. But the phenomenon of photon loss

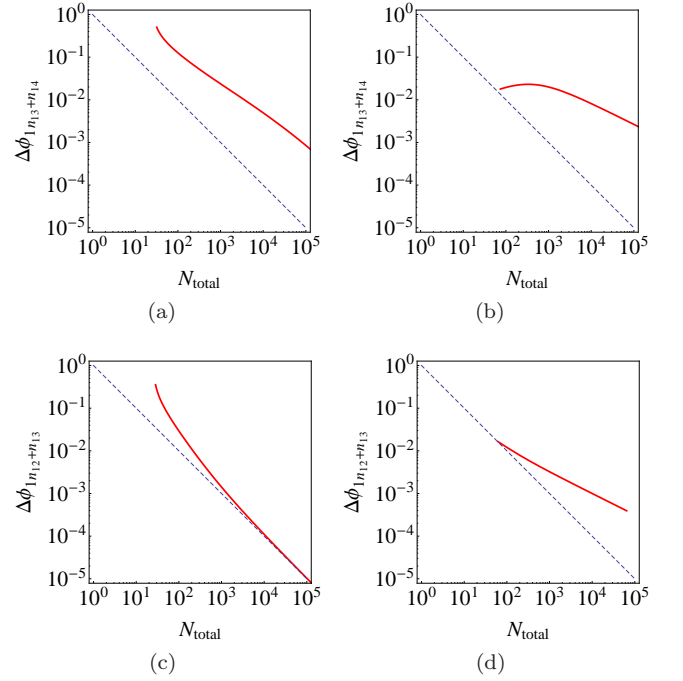


FIG. 8: The phase sensitivities  $(\Delta\phi_1)_{n_{13}+n_{14}}$  and  $(\Delta\phi_1)_{n_{12}+n_{13}}$  as function of the total number  $N_{total}$  in figure (a) and (b) with the coherent light feeding into the first input, and in figure (c) and (d) with the coherent light feeding into the third input, respectively. The dashed blue lines represent the Heisenberg limit. In (a) and (c), set  $|\alpha| = 5$ ,  $\beta_1 = \beta_2$ , and vary the values of  $\beta_1$  and  $\beta_2$ . In (b) and (d), set  $\beta_1 = \beta_2 = 3$  and vary the value of  $|\alpha|$ .

widely exists in practical FWMs.  $SU(1,2)$  interferometers utilizes two more FWMs than  $SU(1,1)$  interferometers, which necessarily introduces higher probability of photon loss. In practical experiments, whether the enhancement of phase sensitivity of  $SU(1,2)$  interferometer is robust to the photon loss error has not been studied now.

In this paper, we show that  $SU(1,2)$  interferometer can enhance the phase sensitivity compared with  $SU(1,1)$  interferometer. This work sheds light on the performance of  $SU(1,n)$  interferometer. We believe by adding more FWMs,  $SU(1,n)$  interferometer can further enhance the phase sensitivity. What's more, in this work, we use the Lie group theoretical method to investigate quantum interferometers. The way of our investigation in this paper can be generalized into any  $SU(m,n)$  interferometer.

In conclusion, this paper uses Lie group and Lie algebra theory to analyze one multi-path nonlinear optical quantum interferometer, denoted by  $SU(1,2)$  interferometer. Eight Hermitian operators spanning Lie algebra  $\mathfrak{su}(1,2)$  are introduced to describe the Hamiltonian of  $SU(1,2)$  interferometer. We calculate and analyze the phase sensitivities in  $SU(1,2)$  interferometer. With all vacuum state inputs, the optimal phase sensitivity is achieved when the sum of the photon numbers in beam 12 and beam 14 is detected. This phase sensitivity of  $SU(1,2)$  inter-

ferometer can achieve the Heisenberg limit and beats the sensitivity of  $SU(1, 1)$  interferometer, because  $SU(1, 2)$  interferometer amplifies the intensity of the input beams twice. As for the case of one coherent state light input, we analyze the optimal detection combination when coherent state light is fed into different input ports. It is shown that the Heisenberg limit can be approached only when the coherent state light is injected into the third input port.

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